Generalised Nash Solution to the Puzzle of 1/2, 3/5, and 2/3 Shares in Sharecropping

Abstract

Stiglitz admits that no widely accepted theory can explicitly determine the commonly observed crop shares of 1/2, 3/5, and 2/3 in sharecropping. This implies that no one signs these contracts in theory and sharecropping does not exist. If the owner of sharecropping farm maximises their expected profits under uncertainty to determine the necessary terms for signing contracts, Stiglitz's puzzle can be solved. For pure sharecropping in between fixed-wage and fixed-rent contracts, the landowner and the sharecropper are joint owners from the risk-sharing partnership perspective and certainly bargain over the yield to maximise their respective profit. Our partnership model endogenises the bargaining powers of both owners, yielding 1/2, 3/5, and 2/3 crop shares according to the generalised Nash solution in the signed sharecropping structures. For accuracy, we employ the strategic game of alternating offers to achieve identical results as the unique Nash perfect equilibrium, thereby justifying our bargaining approach.

JEL Classification: C71, D21, Q12

Keywords: endogenous bargaining powers; generalised Nash solution; puzzle of 1/2, 3/5, and 2/3 crop shares; risk-sharing partnership; sharecropping

1 Introduction

This study aims to solve the puzzle of why most sharecropping contracts have discrete crop shares of 1/2, 3/5, or 2/3 in the real world. If these crop shares cannot be determined *ex-ante*, sharecropping contracts are unlikely to be signed by anyone.

Problem. Stiglitz (1988, 120) admits that the 'agency theory cannot explain why 50-50 is the most commonly observed crop share' because any fraction in $[0,1]$ can be viewed as a qualified crop share. Moreover, the crop share $1/(1+\tau\sigma^2)$ is designed by a risk-neutral landowner for a risk-averse sharecropper (Singh 1989, 37) and has a dimensional error. Suppose the output is measured in kg and the utility u of a sharecropper is a function of their output: The output variance, σ^2, has the dimension kg^2, and the Arrow-Pratt absolute risk aversion coefficient, $\tau = -u''/u'$, has the dimension $1/kg$. Thus, $\tau\sigma^2$ has the dimension kg, and $1+\tau\sigma^2$ violates the 'dimensional homogeneity' where quantities with the same dimensions can be added or subtracted. The dimensional problem also exists in the seminal risk-sharing model developed by Stiglitz (1974, 232), in which an equilibrium crop share is $\frac{1}{3}-\frac{1}{6}kg^2$. This suggests that any pure fraction cannot be solved via the principal-agent approach. If a 50-50 crop share could not be specified, these contracts should not have existed. However, in India, the most common crop share has been 50-50 (Bardhan 1984, 115). In North America, when some input is shared, the 50-50 dominates; when no input is shared, 50-50 falls from first to third place in crop share outcomes following 3/5 or 2/3 (Allen and Lueck 2002, 90). Although various theories seek to explain the stylized facts, none has gained general acceptance (Stiglitz 1989, 22).

Related Literature. Our study follows the work of Bell and Zusman (1976), who first employed a *Nash* bargaining solution to determine the crop share in pure sharecropping. The results of our study, alongside those obtained by Bell and Zusman, imply that sharecropping (particularly *metayage*) is a partnership, which is a sentiment echoed by the research findings of a number of economists (Kikuchi and Hayami 1980; Murrell 1983). Only in a partnership do joint owners need to bargain over the output of their cooperation and share profit related to risk. It is the basis of *Nash* bargaining, as a two-person bargaining situation involves individuals who have an opportunity to collaborate for mutual benefit (Nash 1950). The best result will come from everyone in the group doing what is best for himself and for the group;[1] that is, both the individual rationality and group rationality must be satisfied simultaneously.

Bell and Zusman (1976) achieved only a few crop shares close to 50-50, rather than attaining the complete 50-50 split; this was due to their model focusing solely on determining crop share. Sharecropping actually includes a number of additional variables such as plot size and cost share of the cooperating inputs (Reid 1975). Bell and Zusman (1976) omitted the cooperating input and adopted Cheung's (1968) pioneering approach of averaging the whole land (H) among sharecroppers (m) without considering the land cost,[2] accepting his explanation that $\frac{sq}{h} = \frac{\partial q}{\partial h}$ indicates the following: 'the rent per acre of land equals the marginal product of land in equilibrium' where s is the landowner's crop share, and $h = H/m$ is the plot size of each sharecropper. Factually, sq and $(1 - s)q$ are the fractions of crop revenue q. As m decreases, h increases and $\frac{\partial q}{\partial h}$ decreases due to the diminishing marginal product; consequently, the revenue per acre of land, $\frac{sq}{h}$, also decreases. The curve, $\frac{\partial q}{\partial h}$, lies below the curve, $\frac{sq}{h}$, and neither intersect, $\frac{sq}{h} \neq \frac{\partial q}{\partial h}$. If h decreases, $\frac{\partial q}{\partial h}$ and $\frac{sq}{h}$, increase. Only when $h \to 0$, $\frac{\partial q}{\partial h} \to \infty$, and $\frac{sq}{h} \to \infty$ do $\frac{\partial q}{\partial h}$ and $\frac{sq}{h}$ intersect, $\frac{sq}{h} = \frac{\partial q}{\partial h}$. However, when $h \to 0$, $q \to 0$, neither party will sign such a contract.

Our study also follows Binmore, Rubinstein, and Wolinsky (1986), who demonstrated how to use the power weights α ($0 \leq \alpha \leq 1$) and $1 - \alpha$ for players 1 and 2, respectively, in the generalised *Nash* bargaining model developed by Roth, which captures the differences in player bargaining power. Each bargaining solution is characterised by $\max_{x \in X}(u_1(x_1) - d_1)^{\alpha}(u_2(x_2) - d_2)^{1-\alpha}$, where (d_1, d_2) denote the possible utility gains achievable by both parties when they fail to reach an agreement; (u_1, u_2) represents the potential utilities of both parties among the set of possible divisions, x_i, of a pie or money, M, $x_i \in X$, $X = [(x_1, x_2)|M \geq x_1 \geq 0, x_2 = M - x_1]$. When two players reach an agreement, player 1 gets $u_1 = d_1 + \alpha(M - d_1 - d_2)$ and player 2 gets $u_2 = d_2 + (1 - \alpha)(M - d_1 - d_2)$, which is called the '*Split-The-Difference' Rule* (Muthoo 1999, 36). Binmore, Rubinstein, and Wolinsky (1986) held that, aside from the asymmetry in risk preferences, the α and $1 - \alpha$ must be chosen to reflect the remaining asymmetries in the procedure and the beliefs of each party in their relative bargaining power. This demonstrates that bargaining power is a vague concept dependent upon its sources, interrelations, and effects (Alavoine, Kaplanseren, and Teulon 2014; Eichstädt, Hotait, and Dahlen 2017). Nash (1950) was very cautious about using the term 'bargaining ability' because it suggests that players propagandise each other into misconceptions of the utilities involved. Thus, Nash assumes that the two players are highly rational, each has full knowledge of the other player's preferences, and they have equal bargaining skills, rendering any attempts at deceit meaningless.

However, it should be acknowledged that the preferences of each player are not common knowledge (Young 1993) and cannot be precisely measured. A player is likely to underplay the utility they have gained from a transaction if they deem it beneficial. Note that due to the splitting of the given pie M, the bargaining game appears to be an experiment where the experimenter presents two white mice with a pie and then analyses how the mice split it. In private ownership conditions, people bargain to split the object legitimately owned by themselves. There are ambiguities in locating d_i because of M being unrelated to d_i. One possibility is to identify d_i with the income accruing to the two parties during the dispute. Another possibility is to identify d_i with the income available to the parties if they choose the best available alternative elsewhere (Binmore, Rubinstein, and Wolinsky 1986). The third possibility is to identify d_i with the utility that player i obtains in autarky (Talamàs 2020). If *Nash* bargaining is applied to split the crop coproduced by the landowner and the sharecropper in partnership, all the unrealistic assumptions disappear. We only need to assume that both owners maximise their respective expected profit.

Competitive Land Rental Market. By definition, the primary feature of the competitive land rental market is that one individual is unable to affect the rental rate or price, r (the rent per acre of land). However, some economists regard the landowner's crop share s as a price-like parameter related to a *land lease* (Cheung 1968; Bardhan and Srinivasan 1971; Newbery 1974; Reid 1976). This idea is inconsistent with the following facts. First, the s, such as 1/2, 3/5, and 2/3, are explicitly used to split the crop in sharecropping. We understand the s only by how the landowner and sharecropper use it, rather than by its name, 'rental share'.[3] Second, there is no a land lease market where landowners offer their dimensionless s, and sharecroppers then determine the land size rented. Third, this idea does not allow r and fixed-rent contract to exist in model (Otsuka, Chuma, and Hayami 1992). However, r exists in practice (Cheung 1969, 72).

Stiglitz (1974) introduces the concept of 'utility equivalent contracts' and replaces the price-taking assumption of the competitive model with a 'utility-taking' assumption. Contracts, however, are classified by the individual who claims the residual profit. In an agricultural society, the landowner is the labourer lessee[4] and farm owner in the fixed-wage contract and the tenant is the farm owner and labourer in the fixed-rent contract (Hirshleifer, Glazer, and Hirshleifer 2005, 159). For pure sharecropping in between fixed-wage and fixed-rent contracts, the landowner and the sharecropper are joint owners in the risk-sharing partnership. All farm owners desire profits due to the risk they take (Knight 1921, 36). Considering that a sharecropper takes on a fraction of the risks, they should receive some profits; otherwise, they would sign a fixed-wage contract. A farm labourer (worker, sharecropper, or tenant) receives varying income levels according to the different contract terms. Thus, 'utility equivalent contracts' do not exist for a labourer, and a 'utility-taking' assumption is rendered void.

In the competitive land rental market, everyone must take r rather than a fraction s as given. Farm owners must consider the land cost per acre to maximise profit. Even under uncertainty, profits are the sole criterion by which the competitive market selects survivors (Alchian 1950).

Our model is the first risk-sharing partnership in which a landowner and a sharecropper can simultaneously maximise their respective expected profit. Compared to Bell and Zusman's (1976) model, our model endogenises the bargaining powers of two players and obtains the crop shares of 1/2, 3/5, and 2/3 as the *Nash* bargaining solution. And our model can explain why 50-50 is the dominant crop share and why labour terms are never found in sharecropping.

The article proceeds as follows. Section 2 concerns the method and sharecropping structures, which presents how the dominant crop share (1/2) and other crop shares (3/5 and 2/3) are derived. Section 3 presents the concluding remarks.

2 Method and the Signed Sharecropping Structures

The contractual environment is provided as follows:

(i) Output and input markets are competitive. In a scarce economy, the land rental rate $r > 0$ (price unit/area unit*time unit), wage rate $w > 0$ (price unit/labour unit*time unit)[5], the seed or fertiliser price $\theta > 0$ (price unit/kg), and output price (normalised as one price unit/kg) are common knowledge and provided exogenously. The market guarantees full employment.

(ii) The agricultural economy consists of two typical individuals: a landowner who possesses their own homogeneous land H (measured in acre), and a landless labourer who supplies their own identical labour t (measured in labour unit). Either of these parties can sell their own identical seeds or fertiliser [6] f or buy them in the market.

(iii) The fixed wage, fixed rent, and pure sharecropping contract are freely chosen by all; they have the same contracting cost[7] and timeline of one crop year. A mixed contract is not permitted and the same crop grows under any contract.

(iv) Following Stiglitz's (1974) use of μ as a multiplicative random variable with $E\mu = 1$, $\sigma^2 = E(\mu - 1)^2 > 0$ denotes the weather uncertainty, which is not affected by labour.

(v) Technique is constant. The production function $q = \mu q(h, t, f)$ has the properties $q^{'} > 0$, $q^{''} < 0$, $\frac{\partial^2 q}{\partial h \partial t} > 0$, $\frac{\partial^2 q}{\partial h \partial f} > 0$, $\frac{\partial^2 q}{\partial f \partial t} > 0$, $q(0, t, f) = 0$, $q(h, 0, f) = 0$, and $q(h, t, 0) = 0$. h is the plot size that a farm labourer works on.

We assume that farm owners maximise their expected profits under uncertainty and that no one signs the contract if the terms cannot be specified *ex-ante*. For simplicity, the model only involves a landowner and a labourer on a farm. As $E\mu q(h, t, f) = q(h, t, f)$, we omit the expectation operator E. Farm owner determines the terms by solving the following equation:

$$(1) \qquad \max_{h,t,f} q(h, t, f) - rh - wt - \theta f$$

where $q(h, t, f) - rh - wt - \theta f \geq 0$ always holds;[8] that is, the market values of the individual resources involved in a farm add up to less than the market value of their product.[9] Profit motivates an owner to manage a farm, accruing to the landowner in a fixed-wage farm, and to the tenant in a fixed-rent farm. The owner bears all the costs (including the cost of enforcing contract)[10] and risks.

If a landowner provides h and shares β $(0 \leq \beta \leq 1)$ in f, and a sharecropper provides t and shares $(1-\beta)$ in f to coproduce, then by sharing the expected profit related to risk, they are joint owners in this partnership and jointly determine the necessary terms h, f, β, and the landowner's crop share s $(0 < s < 1)$ in q. $\beta = 0$ indicates that f is entirely supplied by the sharecropper. If $\beta = 1$, f is provided by the landowner. The expected profit in (1) is their shared goal, which equals the total expected revenue $sq(h,t,f) + (1-s)q(h,t,f)$ minus the total input costs $rh + \theta\beta f + wt + \theta(1-\beta)f$. Both maximise the expected profit from the farm.

Solving (1), we get $\frac{\partial q}{\partial h} = r$, $\frac{\partial q}{\partial t} = w$, and $\frac{\partial q}{\partial f} = \theta$, corresponding to Pareto optimal quantities h_2, t_2, and f_2, respectively (see point A in Figures 1, 2, 3). This implies that a landowner pays their worker the fixed wage wt_2 and a tenant pays their landowner the fixed rent rh_2. In pure sharecropping, h_2, t_2, and f_2 characterise the group rationality (*Nash* Pareto-efficiency axiom); that is, each does their best for the farm. The expected profit of a sharecropping farm is thus maximised. However, to sign the contract, both owners need determine the landowner's s in $q(h_2, t_2, f_2)$, and their cost share β in θf_2, characterising the individual rationality that the landowner and the sharecropper do best to maximise their respective expected profits; that is, their respective expected revenue minus the cost of their respective inputs, $sq(h_2,t_2,f_2) - rh_2 - \theta\beta f_2$ and $(1-s)q(h_2,t_2,f_2) - wt_2 - \theta(1-\beta)f_2$. The landowner desires the higher s, and the sharecropper the lower s, given β. This requires both sides to be involved in bargaining. $q(h_2,t_2,f_2)$ is equivalent to the M in the *Nash* program. The utility of landowners and sharecroppers is measured by their expected revenue $sq(h_2,t_2,f_2)$ and $(1-s)q(h_2,t_2,f_2)$, respectively.[11] If they fail to reach an agreement on the s and the β, the landowner rents out h_2

and sells their βf_2 to get the fixed $rh_2 + \theta\beta f_2$; the sharecropper signs the fixed-wage contract and sells their $(1-\beta)f_2$ to get the fixed $wt_2 + \theta(1-\beta)f_2$. These threats are credible in competitive markets. Thus, $rh_2 + \theta\beta f_2$ and $wt_2 + \theta(1-\beta)f_2$ can be viewed as both owner's disagreement payoffs, equivalent to d_1 and d_2 in the *Nash* program.

The *Nash* symmetry axiom says that if $d_1 = d_2$, $u_1^* = u_2^*$, which is equivalent to if $rh_2 + \theta\beta f_2 = wt_2 + \theta(1-\beta)f_2$, $sq(h_2, t_2, f_2) = (1-s)q(h_2, t_2, f_2)$. Solving for s and β, we get $s^* = 1/2$ and $\beta^* = (wt_2 - rh_2 + \theta f_2)/2\theta f_2$.

Proposition 1: In pure sharecropping, the landowner and sharecropper must stipulate the optimal plot size h_2, optimal seed or fertiliser f_2, landowner's cost share $\beta^* = (wt_2 - rh_2 + \theta f_2)/2\theta f_2$ in θf_2, and 50-50 crop share split as per the *Nash* bargaining solution.

This means that $0.5q - rh_2 - \theta\beta^* f_2 = 0.5q - wt_2 - \theta(1-\beta^*)f_2$, which is equivalent to $u_1^* - d_1 = u_2^* - d_2$ in the Nash symmetry axiom. When $d_1 \neq d_2$, the 'regular' *Nash* solution (u_1^*, u_2^*) is obtained by solving $max\,(u_1 - d_1)\,(u_2 - d_2)$. Thus, when $rh_2 + \theta\beta f_2 \neq wt_2 + \theta(1-\beta)f_2$, we solve the equivalent equation:

$$(2) \qquad \max_s [sq(h_2, t_2, f_2) - rh_2 - \theta\beta f_2][(1-s)q(h_2, t_2, f_2) - wt_2 - \theta(1-\beta)f_2]$$

Solving for s given β, we obtain $s = 1/2 + [rh_2 + \theta\beta f_2 - wt_2 - \theta(1-\beta)f_2]/2q$ (Appendix A). However, the s cannot be determined *ex-ante,* unless the unrealised q is already known. Note that the regular *Nash* solution reflects only Nash's assumption that two players have equal bargaining skill. In the real world, the larger the s that the landowner demands, the larger the

$rh_2 + \theta\beta f_2$ relative to the $wt_2 + \theta(1-\beta)f_2$; conversely, the larger the $1-s$ that the sharecropper demands, the larger the $wt_2 + \theta(1-\beta)f_2$ relative to the $rh_2 + \theta\beta f_2$. Thus, we extend Binmore, Rubinstein, and Wolinsky's (1986) approach using the power weights α and $1-\alpha$ to capture the bargaining powers of the landowner and the sharecropper from their disagreement payoffs; that is, $\alpha = (rh_2 + \theta\beta f_2)/(rh_2 + wt_2 + \theta f_2)$ and $1-\alpha = [wt_2 + \theta(1-\beta)f_2)]/(rh_2 + wt_2 + \theta f_2)$. This is also Bell and Zusman's (1976) idea that 'the disagreement payoffs represent the bargaining power of the landlord and sharecropper'. By *Nash*-Roth theorem, the generalised *Nash* solution (u_1^*, u_2^*) is obtained by solving $max\,(u_1 - d_1)^{\alpha}(u_2 - d_2)^{1-\alpha}$, resulting in the following equivalent equation:

$$(3) \qquad \max_{s}[sq - rh_2 - \theta\beta f_2]^{\alpha}[(1-s)q - wt_2 - \theta(1-\beta)f_2]^{1-\alpha}$$

Solving (3), we obtain $s^* = (rh_2 + \theta\beta f_2)/(rh_2 + wt_2 + \theta f_2)$ (Appendix B).

Proposition 2: In pure sharecropping, both sides need to specify the optimal plot size h_2, optimal seed or fertiliser f_2, the landowner's cost-share β in the cost θf_2, and their crop share $s^* = (rh_2 + \theta\beta f_2)/(rh_2 + wt_2 + \theta f_2)$.

The s^* is independent of q. To sign the contract, s^* and β must be explicitly specified; both sides would better set simple and clear-cut s^* and then determine β from $s^* = (rh_2 + \theta\beta f_2)/(rh_2 + wt_2 + \theta f_2)$. If $s^* = 1/2$, $\beta^* = (wt_2 - rh_2 + \theta f_2)/2\theta f_2$, just as with the result in Proposition 1. Here, $\beta \in [0,1]$ plays the role of anti-fluctuator and usually holds even though the h_2, t_2, and f_2 vary with r, w, and θ, respectively. This is why when input f_2 is shared, 1/2 crop share is dominant. If rh_2, wt_2, and θf_2 lead to $\beta \notin [0,1]$, which violates $0 \leq \beta \leq 1$, both sides must shift s^* from 1/2 to 3/5, 2/3, and so on, to ensure $\beta^* \in [0,1]$. If the cost of θf_2 is borne by one side, $s^* \neq 1/2$, unless it happens to be $rh_2 = wt_2 + \theta f_2$ ($\beta^* = 0$) or $rh_2 + \theta f_2 = wt_2$ ($\beta^* = 1$). As the more common case is $rh_2 \neq wt_2 + \theta f_2$ or $rh_2 + \theta f_2 \neq wt_2$, this is why when no input is shared, 50-50 crop share is likely to be dominated by 3/5 or 2/3.

Proposition 3: As long as $\beta \in [0,1]$ holds with the fluctuation of r, w, and θ, and the rh_2, wt_2, and θf_2, the 50-50 crop share should not change in sharecropping. If $\beta^* \notin [0,1]$, the dominant crop share must fall from 1/2 and shift to 3/5, 2/3, and so on, to ensure $\beta^* \in [0,1]$.

Next, we use the strategic game to determine sharecropping terms, where players take turns in proposing offers until they reach an agreement. A sharecropper's offer should maximise their expected profits, subject to the condition that a landowner's expected revenue does not fall below their own inputs costs, and vice versa, which forms each player's individual rationality. The equation of sharecropper and landowner, respectively, can be represented as follows:

(4) $\max_{h,t,f,s,\beta} (1-s)q(h,t,f) - wt - \theta(1-\beta)f$ *s.t.* $sq(h,t,f) \geq rh + \theta\beta f$

(5) $\max_{h,t,f,s,\beta} sq(h,t,f) - rh - \theta\beta f$ *s.t.* $(1-s)q(h,t,f) \geq wt + \theta(1-\beta)f$

First, we solve equation (4). From the associated Lagrangian function $L = (1-s)q(h,t,f) - wt - \theta(1-\beta)f + \lambda[sq(h,t,f) - rh - \theta\beta f]$, assuming an interior solution, the Kuhn-Tucker conditions coincide with the ordinary first-order Lagrangian conditions:

(6) $\frac{\partial L}{\partial h} = (1-s)\frac{\partial q}{\partial h} + \lambda\left(s\frac{\partial q}{\partial h} - r\right) = 0$

(7) $\frac{\partial L}{\partial t} = (1-s)\frac{\partial q}{\partial t} - w + \lambda\left(s\frac{\partial q}{\partial t}\right) = 0$

(8) $\frac{\partial L}{\partial f} = (1-s)\frac{\partial q}{\partial f} - \theta(1-\beta) + \lambda\left(s\frac{\partial q}{\partial f} - \theta\beta\right) = 0$

(9) $\frac{\partial L}{\partial s} =- q(h,t,f) + \lambda q(h,t,f) = 0$

(10) $\frac{\partial L}{\partial \beta} = \theta f + \lambda(-\theta f) = 0$

We also obtain $\frac{\partial q}{\partial h} = r$, $\frac{\partial q}{\partial t} = w$, and $\frac{\partial q}{\partial f} = \theta$ corresponding to h_2, t_2, and f_2, respectively. Note that in (7), given h_2, f_2, and agreed β and s, $s\frac{\partial q}{\partial t} = w - (1-s)\frac{\partial q}{\partial t}$ implies the marginal product revenue of labour that the landowner receives equals the marginal cost of labour that the sharecropper bears. In Figure 2, $s\frac{\partial q}{\partial t}$ is the difference between $\frac{\partial q}{\partial t}$ and $(1-s)\frac{\partial q}{\partial t}$, and $w - (1-s)\frac{\partial q}{\partial t}$ is the difference between the straight line BA and the curve $(1-s)\frac{\partial q}{\partial t}$. The classical models suggest that the sharecropper applies their labour only to point B, where $(1-s)\frac{\partial q}{\partial t} = w$ corresponds to t_1. However, the sharecropper cannot distinguish between $s\frac{\partial q}{\partial t}$ and $(1-s)\frac{\partial q}{\partial t}$. If the sharecropper applies more labour than t_1, they receive the marginal product revenue of labour $(1-s)\frac{\partial q}{\partial t}$ less than the prevailing wage rate w and bear the marginal cost $w - (1-s)\frac{\partial q}{\partial t}$, but $s\frac{\partial q}{\partial t}$ is larger than $w - (1-s)\frac{\partial q}{\partial t}$ because the latter is part of the former. $t > t_1$ will make q increase and then both parties will be better off. Due to $q'' < 0$, $s\frac{\partial q}{\partial t}$ decreases and $w - (1-s)\frac{\partial q}{\partial t}$ increases; hence, they all equal AC at point C where the labour level equals t_2 and the mutual benefit opportunity is exhausted. In (6), $(1-s)\frac{\partial q}{\partial h} = r - s\frac{\partial q}{\partial h}$ implies that the marginal product revenue of land that the sharecropper receives equals the marginal cost of land that the landowner bears, given t_2, f_2, and agreed β and s. The classical models never mention the $(1-s)\frac{\partial q}{\partial h}$ taken by the sharecropper. In Figure 1, $(1-s)\frac{\partial q}{\partial h}$ is the difference between $\frac{\partial q}{\partial h}$ and $s\frac{\partial q}{\partial h}$; $r - s\frac{\partial q}{\partial h}$ is the difference between the r and $s\frac{\partial q}{\partial h}$ beyond point B corresponding to h_1. If the landowner provides land $h > h_1$, $(1-s)\frac{\partial q}{\partial h} > r - s\frac{\partial q}{\partial h}$ and this makes q increase until $(1-s)\frac{\partial q}{\partial h}$ and $r - s\frac{\partial q}{\partial h}$ are all equal to AC at point C where $\frac{\partial q}{\partial h} = r$. Equation (8) appears a bit complicated, so we rearrange it to obtain $s\frac{\partial q}{\partial f} - \theta\beta = \theta(1-\beta) - (1-s)\frac{\partial q}{\partial f}$. Given h_2, t_2, and an agreed β and s, $s\frac{\partial q}{\partial f} - \theta\beta$ is the net marginal product revenue that the landowner receives and $\theta(1-\beta) - (1-s)\frac{\partial q}{\partial f}$ is the net marginal cost that the sharecropper bears. In Figure 3, the sharecropper's marginal product revenue of the cooperating input f is $(1-s)\frac{\partial q}{\partial f}$, and the landowner's marginal product revenue of f, $s\frac{\partial q}{\partial f}$, is the difference between $\frac{\partial q}{\partial f}$ and $(1-s)\frac{\partial q}{\partial f}$. If the sharecropper chooses the quantity of f at point D where $(1-s)\frac{\partial q}{\partial f} = \theta(1-\beta)$ corresponding to f_1, the landowner's $s\frac{\partial q}{\partial f} - \theta\beta > 0$. If the sharecropper chooses $f > f_1$, $s\frac{\partial q}{\partial f} - \theta\beta > \theta(1-\beta) - (1-s)\frac{\partial q}{\partial f}$. As $\theta(1-\beta) - (1-s)\frac{\partial q}{\partial f}$ is part of $s\frac{\partial q}{\partial f} - \theta\beta$, as f increases, $s\frac{\partial q}{\partial f} - \theta\beta$ decreases, and $\theta(1-\beta) - (1-s)\frac{\partial q}{\partial f}$ increases because $q'' < 0$. This makes q increase until $s\frac{\partial q}{\partial f} - \theta\beta$ and $\theta(1-\beta) - (1-s)\frac{\partial q}{\partial f}$ are all equal to EC at point C where $\frac{\partial q}{\partial f} = \theta$ corresponding to f_2.

To maximise $(1-s)q(h,t,f) - wt - \theta(1-\beta)f$, the sharecropper derives their marginal product revenue of labour from $(1-s)q(h,t,f)$

$$(11) \qquad \frac{d(1-s)q}{dt} = (1-s)\left(\frac{\partial q}{\partial h}\frac{dh}{dt} + \frac{\partial q}{\partial t} + \frac{\partial q}{\partial f}\frac{df}{dt}\right) = w + \theta(1-\beta)\frac{df}{dt}$$

where $(1-s)\frac{\partial q}{\partial t} = w$ corresponding to t_1 (Figure 2) does not hold unless $\frac{\partial q}{\partial h} = 0$ and $\frac{\partial q}{\partial f} = 0$, which is impossible in a scarce economy.[12] The marginal labour product revenue of the sharecropper depends on h, t, f, and desired s and β. The sharecropper must choose h_2, t_2, and f_2; that is, $\frac{\partial q}{\partial h} = r$, $\frac{\partial q}{\partial t} = w$, and $\frac{\partial q}{\partial f} = \theta$ to make equation (11) hold. Substituting these into equation (11), we obtain the following:

$$(12) \qquad (1-s)\left(r\frac{dh}{dt} + w + \theta\frac{df}{dt}\right) = w + \theta(1-\beta)\frac{df}{dt}$$

Solving for s given β using the closed intervals $[0, h_2]$, $[0, t_2]$, and $[0, f_2]$, respectively, we obtain the following:

$$(13) \qquad (1-s)\left(\int_0^{h_2} r dh + \int_0^{t_2} w dt + \int_0^{f_2} \theta df\right) = \int_0^{t_2} w dt + \int_0^{f_2} \theta(1-\beta) df$$

$s^* = (rh_2 + \theta\beta f_2)/(rh_2 + wt_2 + \theta f_2)$. Thus, the sharecropper offers h_2, t_2, f_2, β, and s^* at each turn in the contract negotiation.

Now, we solve equation (5). From the associated Lagrangian function $\zeta = sq(h,t,f) - rh - \theta\beta f + \eta[(1-s)q(h,t,f) - wt - \theta(1-\beta)f]$, we obtain the same solutions $\frac{\partial q}{\partial h} = r$, $\frac{\partial q}{\partial t} = w$, and

$\frac{\partial q}{\partial f} = \theta$, corresponding to h_2, t_2 and, f_2 (Appendix C), respectively. Since all the geometric analyses of the first-order conditions are similar to those of equation (4), we do not reiterate them here. To maximise $sq(h,t,f) - rh - \theta\beta f$, the landowner derives their land marginal product revenue from $sq(h,t,f)$.

(14) $$\frac{dsq}{dh} = s\left(\frac{\partial q}{\partial h} + \frac{\partial q}{\partial t}\frac{dt}{dh} + \frac{\partial q}{\partial f}\frac{df}{dh}\right) = r + \theta\beta\frac{df}{dh}$$

Similarly, the rational landowner chooses h_2, t_2, and f_2; that is, $\frac{\partial q}{\partial h} = r$, $\frac{\partial q}{\partial t} = w$, and $\frac{\partial q}{\partial f} = 0$ to make (14) hold. Substituting these into (14), we obtain the following:

(15) $$s\left(r + w\frac{dt}{dh} + \theta\frac{df}{dh}\right) = r + \theta\beta\frac{df}{dh}$$

Solving for s given β, using the intervals $[0,h_2]$, $[0,t_2]$, and $[0,f_2]$, we obtain $s^* = (rh_2 + \theta\beta f_2)/(rh_2 + wt_2 + \theta f_2)$ from the following integral equation:

(16) $$s\left(\int_0^{h_2} r dh + \int_0^{t_2} w dt + \int_0^{f_2} \theta df\right) = \int_0^{h_2} r dh + \int_0^{f_2} \theta\beta df$$

Thus, the landowner offers h_2, t_2, f_2, β, and s^* at each turn. This means that h_2, t_2, f_2, β, and s^* are both sides' best choice and thereby comprise the unique *Nash* perfect equilibrium, regardless of who offers first. This proves *Nash*'s idea that 'the two approaches to the problem, via the negotiation, or axioms, are complementary; each helps to justify and clarify the other'.[13]

When signing the contract, both need not stipulate all the optimal quantities. Suppose that both sides only specify h_2, f_2, β, and s^* in the contract, if the sharecropper provided labour t_2 *ex-post*, both sides would obtain their respective optimal expected revenue (Appendix D):

$$(17)\ \ (1-s^*)q(h_2,t_2,f_2) = wt_2 + \theta(1-\beta)f_2 + (1-s^*)[q(h_2,t_2,f_2) - rh_2 - wt_2 - \theta f_2]$$

$$(18)\ \ s^*q(h_2,t_2,f_2) = rh_2 + \theta\beta f_2 + s^*[q(h_2,t_2,f_2) - rh_2 - wt_2 - \theta f_2]$$

Here, (17) and (18) are equivalent to the *Split-the-Difference* Rule, $u_2 = d_2 + (1-\alpha)(M - d_1 - d_2)$ and $u_1 = d_1 + \alpha(M - d_1 - d_2)$. If the sharecropper provided $t_1 < t_2$, $(1-s^*)q(h_2,t_1,f_2) < (1-s^*)q(h_2,t_2,f_2)$, $s^*q(h_2,t_1,f_2) < s^*q(h_2,t_2,f_2)$ due to $q^{'} > 0$, both would suffer losses. Thus, h_2, f_2, β, and s^* are optimal not only *ex-ante* but also *ex-post;* that is, they are dynamically consistent.

Proposition 4: If h_2, f_2, β, and s^* are specified in pure sharecropping, the landowner and the sharecropper will voluntarily sign the contract. Optimal labour t_2 can be self-enforced *ex-post* by the sharecropper and need not be specified in the contract.

This is why the labour terms are never found in sharecropping. As h_2, f_2, β, and s^* can maximise the expected profit of each owner and the farm, landowner and sharecropper will sign the contracts of (h_2, f_2, β^*, 1/2), (h_2, f_2, β^*, 3/5), and (h_2, f_2, β^*, 2/3) provided $0 \le \beta^* \le 1$ is achieved according to $s = (rh_2 + \theta\beta^* f_2)/(rh_2 + wt_2 + \theta f_2)$, s = 1/2, 3/5, 2/3.

3 Concluding Remarks

Our model does not follow the prevailing Cheung's (1968) or Stiglitz's (1974) paradigm as their landowner does not pursue profits. We cannot identify the farm owner based on the structure of residual claims, nor can we ascertain whether the landowner can achieve the maximum revenue or utility in equilibrium. In mathematical analysis, regardless of finding the maximum or minimum with a condition, the Lagrange multiplier method is the only available approach. If we replace $max\, m\, sq(h,t)$ with $min\, m\, sq(h,t)$ in Cheung's (1968) model, all the first-order conditions remain unchanged, and $h = H/m \rightarrow 0$ still makes $\frac{sq}{h} = \frac{\partial q}{\partial h}$ hold in equilibrium. When $h \rightarrow 0$, $msq(h,t) \rightarrow 0$ is the landowner's minimum rather than maximum revenue. This is because maxima and minima occur at critical points where all partial derivatives are zero. Only via diminishing marginal returns and the constant marginal cost of each input, can we discover the maximum from critical points.

Our model follows *Nash*'s bargaining paradigm because it requires both the individual rationality of each player and the group rationality to be satisfied. This corresponds to maximising the expected profits of each individual and the sharecropping farm in a risk-sharing partnership. When signing sharecropping contracts, both owners must bargain over the maximum expected revenue $q(h_2, t_2, f_2)$. The generalised Nash solution $s^* = (rh_2 + \theta\beta f_2)/(rh_2 + wt_2 + \theta f_2)$ obtained makes the *Nash* symmetry solution 50-50 become the dominant outcome among the commonly observed crop share 1/2, 3/5, and 2/3. If the landowner and the sharecropper provide the same cost of inputs, $rh_2 + \theta\beta f_2 = wt_2 + \theta(1-\beta)f_2$, they have equal bargaining power and equal expected profit $0.5(q - rh_2 - wt_2 - \theta f_2)$; if $rh_2 + \theta\beta f_2 \neq wt_2 + \theta(1-\beta)f_2$, they have unequal bargaining power and unequal expected profit, $s^*(q - rh_2 - wt_2 - \theta f_2)$ and $(1-s^*)(q - rh_2 - wt_2 - \theta f_2)$, respectively (Appendix D). As the landowner and sharecropper receive their mutually agreed crop share of the expected profit, they voluntarily sign this risk-sharing partnership. However, Stiglitz's (1974) concept of 'utility equivalent contracts' implies that sharecropping always exists regardless of crop share and expected profit. In the model with a cooperating input involved in a sharecropping farm, a sharecropper's expected revenue (17) is higher than the fixed wage wt_2 of a worker. It shows that a farm labourer has different utility in different contract.

The limitation of our study is that sharecropping is a unique example involving the bargaining problem in a risk-sharing partnership; we cannot guarantee that our method of endogenising bargaining power is applicable to the generalised case. This is the future direction of the *Nash* bargaining research.

Appendix A

$$N = [sq(h_2, t_2, f_2) - rh_2 - \theta\beta f_2][(1-s)q(h_2, t_2, f_2) - wt_2 - \theta(1-\beta)f_2]$$

$$\frac{d\ln N}{ds} = \frac{q(h_2,t_2,f_2)}{sq - rh_2 - \theta\beta f_2} - \frac{q(h_2,t_2,f_2)}{(1-s)q - wt_2 - \theta(1-\beta)f_2} = 0$$

$$sq(h_2, t_2, f_2) - rh_2 - \theta\beta f_2 = (1-s)q(h_2, t_2, f_2) - wt_2 - \theta(1-\beta)f_2$$

$$s = 1/2 + [rh_2 + \theta\beta f_2 - wt_2 - \theta(1-\beta)f_2]/2q$$

Appendix B

$$\alpha = (rh_2 + \theta\beta f_2)/(rh_2 + wt_2 + \theta f_2)$$

$$N = [sq - rh_2 - \theta\beta f_2]^{\alpha}[(1-s)q - wt_2 - \theta(1-\beta)f_2]^{1-\alpha}$$

$$\frac{dLnN}{ds} = \frac{rh_2 + \theta\beta f_2}{rh_2 + wt_2 + \theta f_2} \bullet \frac{q}{sq - rh_2 - \theta\beta f_2} + \frac{wt_2 + \theta(1-\beta)f_2}{rh_2 + wt_2 + \theta f_2} \bullet \frac{-q}{(1-s)q - wt_2 - \theta(1-\beta)f_2} = 0$$

$$s^* = \frac{rh_2 + \theta\beta f_2}{rh_2 + wt_2 + \theta f_2}$$

Appendix C

$$\zeta = sq(h,t,f) - rh - \theta\beta f + \eta[(1-s)q(h,t,f) - wt - \theta(1-\beta)f]$$

$$\frac{\partial\xi}{\partial h} = s\frac{\partial q}{\partial h} - r + \eta\left[(1-s)\frac{\partial q}{\partial h} = 0\right]$$

$$\frac{\partial\xi}{\partial t} = s\frac{\partial q}{\partial t} + \eta\left[(1-s)\frac{\partial q}{\partial t} - w\right] = 0$$

$$\frac{\partial\xi}{\partial f} = s\frac{\partial q}{\partial f} - \theta\beta + \eta\left[(1-s)\frac{\partial q}{\partial f} - \theta(1-\beta)\right] = 0$$

$$\frac{\partial\xi}{\partial s} = q(h,t,f) + \eta[-q(h,t,f)] = 0$$

$$\frac{\partial\xi}{\partial \beta} =- \theta f + \eta[\theta f] = 0$$

Thus, we obtain $\eta = 1, \frac{\partial q}{\partial h} = r, \frac{\partial q}{\partial t} = w, \frac{\partial q}{\partial f} = \theta$.

Appendix D

Note that $s^* = (rh_2 + \theta\beta f_2)/(rh_2 + wt_2 + \theta\beta f_2)$ and $1 - s^* = [wt_2 + \theta(1-\beta)f_2]/(rh_2 + wt_2 + \theta\beta f_2)$, $s^* \neq 1 - s^*$ if $rh_2 + \theta\beta f_2 \neq wt_2 + \theta(1-\beta)f_2$. Thus, we obtain the expected profits of the sharecropper and the landowner, respectively, if h_2, t_2, f_2, and β are given.

$$(1-s^*)q(h_2,t_2,f_2) - wt_2 - \theta(1-\beta)f_2 = (1-s^*)[q(h_2,t_2,f_2) - rh_2 - wt_2 - \theta f_2]$$

$$s^*q(h_2,t_2,f_2) - rh_2 - \theta\beta f_2 = s^*[q(h_2,t_2,f_2) - rh_2 - wt_2 - \theta f_2]$$

Thus, we derive (17) and (18):

$$(1-s^*)q(h_2,t_2,f_2) = wt_2 + \theta(1-\beta)f_2 + (1-s^*)[q(h_2,t_2,f_2) - rh_2 - wt_2 - \theta f_2]$$

$$s^*q(h_2,t_2,f_2) = rh_2 + \theta\beta f_2 + s^*[q(h_2,t_2,f_2) - rh_2 - wt_2 - \theta f_2]$$

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Endnotes

[1] This is a dialogue from the biographical film 'A Beautiful Mind' on John Nash, it represents the spirit of the *Nash* axiomatic approach,

[2] Stigler informed Cheung that rent is a cost of production (Cheung 1969, 42: footnote 38).

[3] Share-tenancy is not tenancy, just as the peanut is not a nut. Peanuts belong to the botanical family *Fabaceae* (the bean family).

[4] Workers are rented. It is not possible to buy the labour effort of workers (Barzel 2003, 49).

[5] Because the rent of land and the wage of labour depends on the duration of a contract, the rent per acre r and wage per unit of labour w are time-related.

[6] Seeds and fertiliser are indispensable to agricultural production.

[7] The contracting cost does not include the enforcement costs. We are only concerned with contract structures that they voluntarily sign.

[8] Jehle and Reny (2011, 221) rule out the constant returns-to-scale, to ensure the uniqueness of profit maximisation.

[9] See Alchian and Woodward (1987), who explained that a firm is a production unit that does not consume all of its output and is economically viable because of the gains in productivity from specialisation.

[10] See Alchian and Demsetz (1972).

[11] The utility and production function have the same properties: continuous, strictly increasing, and strictly quasiconcave on R_+^n.

[12] See Johnson (1950) and Reid (1976) believed $\frac{\partial q}{\partial h} = 0$ to be a paradox in a scarce economy.

[13] See *Nash* (1953, 29).

Figure Legends

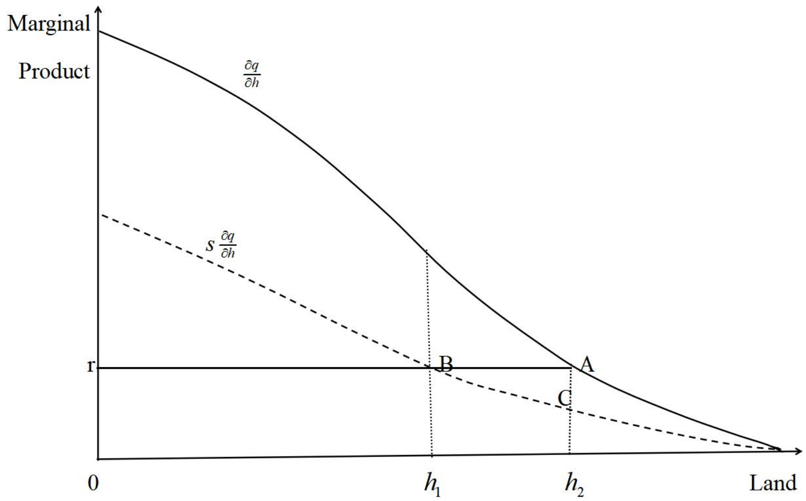

Figure 1 Determination of land size in pure sharecropping of a landowner and sharecropper

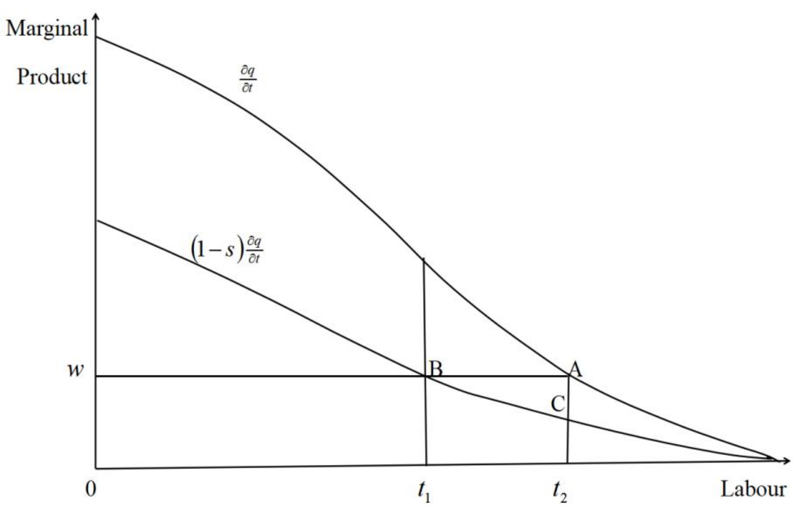

Figure 2 Determination of labour level in pure sharecropping of a landowner and sharecropper

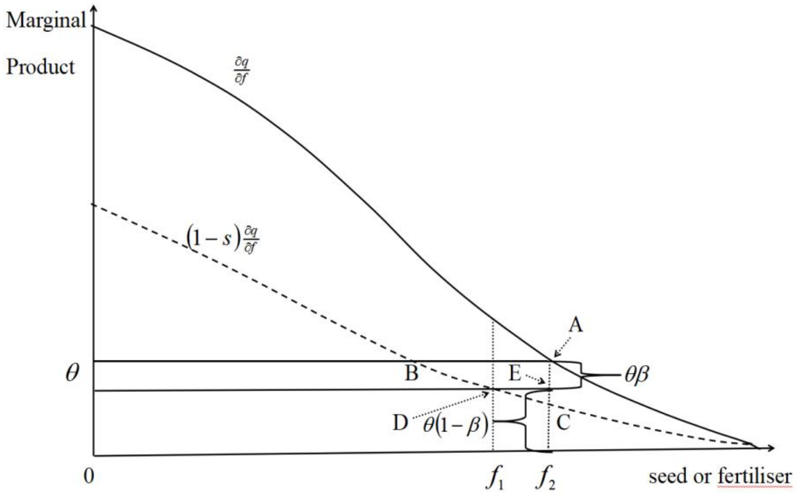

Figure 3 Determination of seed or fertilizer in pure sharecropping of a landowner and sharecropper